Math 31 - Homework 2

Due Friday, July 6

Easy

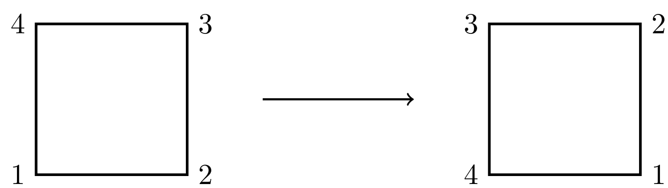
1. Recall from the last homework assignment that if $*$ is a binary operation on a set S, an element x of S is an **idempotent** if $x * x = x$. Prove that a group has exactly one idempotent element.

2. [Herstein, Section 2.1 #5] Let D_4 be the 4th dihedral group, which consists of symmetries of the square. Let $r \in D_4$ denote counterclockwise rotation by $90°$, and let m denote reflection across the vertical axis of the square. Show that

$$rm = mr^{-1}.$$

Conclude that D_4 is a nonabelian group of order 8.

3. We mentioned in class that elements of D_n can be thought of as permutations of the vertices of the regular n-gon. For example, the rotation r of the square mentioned in the last problem,



can be identified with the permutation

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

Write the reflection m as a permutation $\mu \in S_4$, and compute the product $\rho\mu$ in S_4. Then compute $rm \in D_4$, and write it as a permutation σ. Check that $\sigma = \rho\mu$. (In other words, this identification of symmetries of the square with permutations respects the group operations.)

4. Determine whether each of the following subsets is a subgroup of the given group. If not, state which of the subgroup axioms fails.

(a) The set of real numbers $\mathbb{R}$, viewed as a subset of the complex numbers $\mathbb{C}$ (under addition).

(b) The set $\pi\mathbb{Q}$ of rational multiples of π, as a subset of $\mathbb{R}$.

(c) The set of $n \times n$ matrices with determinant 2, as a subset of $\mathrm{GL_n}(\mathbb{R})$.

(d) The set $\{i, m_1, m_2, m_3\} \subset D_3$ of reflections of the equilateral triangle, along with the identity transformation.

5. [Herstein, Section 2.3 #1] Let G be a group. If H and K are subgroups of G, show that $H \cap K$ is also a subgroup of G.

Medium

6. [Herstein, Section 1.4 #2] Let S be a set, and recall that $A(S)$ is the group consisting of the bijections from S to itself, with the binary operation given by composition of functions. Given $s_1 \in S$, define

$$H = \{f \in A(S) : f(s_1) = s_1\}.$$

Show that:

(a) $i \in H$. (Here i denotes the identity function on S.)

(b) If $f, g \in H$, then $fg \in H$. (Note that fg means $f \circ g$.)

(c) If $f \in H$, then $f^{-1} \in H$.

Looking at these three properties, what have you proven about H?

7. [Herstein, Section 2.1 #18] If G is a finite group of *even* order, show that there is an element $a \in G$ (with $a \neq e$) such that $a = a^{-1}$. [Hint: You may want to think about the fact that for any group element x, $(x^{-1})^{-1} = x$.]

8. [Herstein, Section 2.3 #4] Let G be a group. Define

$$Z(G) = \{a \in G : ax = xa \text{ for all } x \in G\}.$$

In other words, the elements of $Z(G)$ are exactly those which commute with *every* element of G. Prove that $Z(G)$ is a subgroup of G, called the **center** of G.

9. Show that if H and K are subgroups of an abelian group G, then

$$\{hk : h \in H \text{ and } k \in K\}$$

is a subgroup of G.

Hard

10. [Herstein, Section 2.3 # 26] Let G be a group, and let H be a subgroup of G. For a fixed $a \in G$, define

$$Ha = \{ha : h \in H\}.$$

Show that, given any $a, b \in G$, we have either $Ha = Hb$ or $Ha \cap Hb = \varnothing$.